The second order cohomology and cyclic cohomology groups of some commutative semigroup algebra

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Abstract: In this paper, we shall reformulate the second order cohomology and cyclic cohomology groups of some commutative semigroup algebras.

Keywords: semigroup, cohomology and cyclic cohomology groups, semigroup algebra.

Introduction:

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$- bimodule, in particular for $X = \mathcal{A}^*$ is a Banach $\mathcal{A}$- bimodule, which is called the dual module of $\mathcal{A}$, and also $\mathcal{A}^*$ is a unit-linked bimodule when $\mathcal{A}$ is unital.

In their article [2], H. G. Dales and J. Duncan established some nice results about $\mathcal{H}^2(\mathcal{A}, X)$, where $\mathcal{A} = \ell^1(S)$, the semigroup algebra of S for some certain semigroups S such as $S = \mathbb{Z}_+$. Indeed, it was proved that $\mathcal{H}^2(\mathcal{A}, \mathcal{A}^*) = \{0\}$ for $\mathcal{A} = \ell^1(S)$ where $S = \mathbb{Z}_+$.

In [3], F. Gourdeau, A. Pourabbas, and M. White investigated the second–order cohomology group of certain semigroup algebras. They proved that $\mathcal{H}^2(\ell^1(S^1), \ell^1(S^1)^*)$ is a Banach space whenever S^1 is any Rees semigroup with identity adjoined.

Let S be the semigroup $T_n = \{e, a, a^2, \dots, a^{n-1}, a^n = o\}$ for $n \in \mathbb{N}$ with $n \geq 2$. We use e for the identity of S We note that T_n is finite, commutative, 0-cancellative, $nil^\sharp$-semigroup which was introduced in [4].

From now on we fix the notation $\mathcal{A}_n$ for the semigroup algebra $\ell^1(T_n)$. In this paper we shall reformulate the second order cohomology and cyclic cohomology groups $\mathcal{H}^2(\mathcal{A}_n, \mathcal{A}_n^*)$ and $\mathcal{HC}^2(\mathcal{A}_n, \mathcal{A}_n^*)$ for the semigroup algebra $\mathcal{A}_n$.

In the next three sections, we recommend the reader to follow [1] for more information.

2 Cohomology of algebras

Let $\mathcal{A}$ be an algebra, and let X be an $\mathcal{A}$-bimodule. For $n \in \mathbb{N}$, recall that an n-linear map $T: \mathcal{A}^n \to X$ is an n-*cochain* and that $\mathcal{L}^n(\mathcal{A}, X)$ is the space of n-linear maps from $\mathcal{A} \times \cdots \times \mathcal{A}$ to X.

Definition 2.1 *Let $n \in \mathbb{N}$. We define the map $\delta^n: \mathcal{L}^n(\mathcal{A}, X) \to \mathcal{L}^{n+1}(\mathcal{A}, X)$ by the formula*

$$(\delta^n T)(a_1, \dots, a_{n+1}) = a_1 \cdot T(a_2, \dots, a_{n+1}) + \sum_{k=1}^{n} (-1)^k T(a_1, \dots, a_{k-1}, a_k a_{k+1}, \dots, a_{n+1}) + (-1)^{n+1} T(a_1, \dots, a_n) \cdot a_{n+1}, \quad (2.1)$$

where $a_1, \dots, a_{n+1} \in \mathcal{A}$ and $T \in \mathcal{L}^n(\mathcal{A}, X)$. We also define $\delta^0: X \to \mathcal{L}(\mathcal{A}, X)$ by $\delta^0(x) = \delta_x \quad (x \in X)$.

Take $n \in \mathbb{N}$. Clearly $\delta^n T \in \mathcal{L}^{n+1}(\mathcal{A}, X)$ for each $T \in \mathcal{L}^n(\mathcal{A}, X)$ and each δ^n is linear. It can be seen by a tedious calculations that $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{N}$. An n-cochain T is an n-*cocycle* if $\delta^n T = 0$, and T is an n-*coboundary* if there is a linear map $Q \in \mathcal{L}^{n-1}(\mathcal{A}, X)$ such that $T = \delta^{n-1} Q$. The linear space of all n-cocycles is denoted by $Z^n(\mathcal{A}, X)$, and the linear space of all n-coboundaries is denoted by $N^n(\mathcal{A}, X)$. Since

$\delta^n \circ \delta^{n-1} = 0$ for all $n \in \mathbb{N}$, the space $N^n(\mathcal{A}, X)$ is a subspace of $Z^n(\mathcal{A}, X)$.

Definition 2.2 *The n^{th}-cohomology group of $\mathcal{A}$ with coefficients in X is defined by*

$$H^n(\mathcal{A}, X) = Z^n(\mathcal{A}, X)/N^n(\mathcal{A}, X).$$

In the additional case where $n = 0$, we set

$$Z^0(\mathcal{A}, X) = ker\,\delta^0 = \{x \in X: a \cdot x = x \cdot a \quad (a \in \mathcal{A})\}$$

and $H^0(\mathcal{A}, X) = Z^0(\mathcal{A}, X)$.

Given $T \in Z^n(\mathcal{A}, X)$, we shall sometimes write $[T]$ for the element of $H^n(\mathcal{A}, X)$ determined by T.

For example, a linear map $D \in \mathcal{L}(\mathcal{A}, X)$ is 1-cocycle if and only if it is a derivation and a 1-coboundary if and only if it is inner.

A map $T \in \mathcal{L}^2(\mathcal{A}, X)$ is a 2-*cocycle* if and only if it satisfies the equation

$$a \cdot T(b, c) - T(ab, c) + T(a, bc) - T(a, b) \cdot c = 0 \quad (a, b, c \in \mathcal{A}). \quad (2.2)$$

Now take a map $Q \in \mathcal{L}(\mathcal{A}, X)$. Then

$$(\delta^1 Q)(x, y) = x \cdot Q(y) - Q(xy) + Q(x) \cdot y \quad (x, y \in \mathcal{A}), \quad (2.3)$$

Clearly $\delta^1 Q \in \mathcal{L}^2(\mathcal{A}, X)$. Each such bilinear map $\delta^1 Q$ is easily checked to be a 2-cocycle.

3 Cohomology of Banach algebras

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. For $T \in \mathcal{B}^n(\mathcal{A}, X)$, we have $\delta^n T \in \mathcal{B}^{n+1}(\mathcal{A}, X)$ and $\delta^n: \mathcal{B}^n(\mathcal{A}, X) \to \mathcal{B}^{n+1}(\mathcal{A}, X)$ is a continuous linear map.

An n-cochain T is a *continuous n-coboundary* if there is a bounded linear map $Q \in \mathcal{B}^n(\mathcal{A}, X)$ such that $T = \delta^n Q$. The linear space of all continuous n-cocycles is denoted by $\mathcal{Z}^n(\mathcal{A}, X)$, and linear space of all continuous n-coboundaries is denoted by $\mathcal{N}^n(\mathcal{A}, X)$. Clearly $\mathcal{Z}^n(\mathcal{A}, X)$ is a closed subspace of $\mathcal{B}^n(\mathcal{A}, X)$ and $\mathcal{N}^n(\mathcal{A}, X)$ is a subspace of $\mathcal{Z}^n(\mathcal{A}, X)$; it is not necessarily closed.

Definition 3.1 *Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. Then the n^{th}-cohomology group of $\mathcal{A}$ with coefficients in X is defined by*

$$\mathcal{H}^n(\mathcal{A}, X) = \mathcal{Z}^n(\mathcal{A}, X)/\mathcal{N}^n(\mathcal{A}, X).$$

The space $\mathcal{H}^n(\mathcal{A}, X)$ is a semi-normed space for the quotient seminorm; it is a Banach space whenever $\mathcal{N}^n(\mathcal{A}, X)$ is closed in $\mathcal{B}^n(\mathcal{A}, X)$.

Definition 3.2 *Let $\mathcal{A}$ be a Banach algebra. A trace on $\mathcal{A}$ is an element T of $\mathcal{A}^*$ such that $T(ab) = T(ba)$ for all $a, b \in \mathcal{A}$. The set of all traces on $\mathcal{A}$ is denoted by $\mathcal{A}^{tr}$.*

We set

$$\mathcal{H}^0(\mathcal{A}, X) = ker\,\delta^0 = \{x \in X: a \cdot x = x \cdot a \quad (a \in \mathcal{A})\}.$$

It is clear that

$$\mathcal{H}^0(\mathcal{A}, \mathcal{A}^*) = \mathcal{A}^{tr} \quad .(3.1)$$

Remark 3.3 *We recall another notation: we define*

$$\widetilde{N}^2(\mathcal{A}, X) = N^2(\mathcal{A}, X) \cap \mathcal{Z}^2(\mathcal{A}, X),$$

and then we define

$$\widetilde{H}^2(\mathcal{A}, X) = \mathcal{Z}^2(\mathcal{A}, X)/\widetilde{N}^2(\mathcal{A}, X).$$

Thus $H^2(\mathcal{A}, X) = \{0\}$ means that, for each $T \in Z^2(\mathcal{A}, X)$, there exists $Q \in \mathcal{L}(\mathcal{A}, X)$, not necessarily continuous, such that $T = \delta^1 Q$, whereas $\mathcal{H}^2(\mathcal{A}, X) = \{0\}$ means that, for each $T \in \mathcal{Z}^2(\mathcal{A}, X)$ there exists a continuous linear map $Q \in \mathcal{B}(\mathcal{A}, X)$ such that $T = \delta^1 Q$. In contrast, $\widetilde{H}^2(\mathcal{A}, X) = \{0\}$

means that, given $T \in \mathcal{Z}^2(\mathcal{A}, X)$, there exists a linear map $Q \in \mathcal{L}(\mathcal{A}, X)$ such that $T = \delta^1 Q$. In fact the vanishing of the continuous second-order cohomology implies that $\widetilde{H}^2(\mathcal{A}, X) = \{0\}$. In our initial cases, our algebra $\mathcal{A}$ will be finite-dimensional, so that there is no difference between $H^2(\mathcal{A}, X)$, $\mathcal{H}^2(\mathcal{A}, X)$, and $\widetilde{H}^2(\mathcal{A}, X)$.

4 Cyclic cohomology of Banach algebras

Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{A}^*$ be its dual bimodule. Take $n \in \mathbb{N}$. An n-cochain $T \in \mathcal{B}^n(\mathcal{A}, \mathcal{A}^*)$ is *cyclic* if it satisfies the equation:

$$T(a_1, \dots, a_n)(a_0) = (-1)^n T(a_0, a_1, \dots, a_{n-1})(a_n) \quad (4.1)$$

whenever $a_0, a_1, \dots, a_n \in \mathcal{A}$.

For example, a linear map $T: \mathcal{A} \to \mathcal{A}^*$ is cyclic if $T(b)(a) = (-1)T(a)(b)$ for all $a, b \in \mathcal{A}$; in other words,

$$\langle a, T(b) \rangle + \langle b, T(a) \rangle = 0 \quad (a, b \in \mathcal{A}). \quad (4.2)$$

In particular,

$$\langle a, T(a) \rangle = 0 \quad (a \in \mathcal{A}), \quad (4.3)$$

and this condition is sufficient to ensure that T is cyclic.

A bounded bilinear 2-cochain $T: \mathcal{A} \times \mathcal{A} \to \mathcal{A}^*$ is cyclic if

$$\langle a, T(b, c) \rangle = \langle c, T(a, b) \rangle \quad (a, b, c \in \mathcal{A}). \quad (4.4)$$

The linear space of all cyclic n-cochains is denoted by $\mathcal{CC}^n(\mathcal{A})$ for $n \geq 1$, and we set $\mathcal{CC}^0(\mathcal{A}) = \mathcal{A}^*$.

It can be seen that the map δ^n maps a cyclic n-cochain to a cyclic one for $n \geq 0$ (see for example page 450 in [5]), so that the cyclic n-cochains $\mathcal{CC}^n((\mathcal{A}), \delta^n)$ form a subcomplex of $\mathcal{B}^n((\mathcal{A}, \mathcal{A}^*), \delta^n)$ and the *differentials* of this complex or its coboundaries are denoted by

$$\delta c^n : \mathcal{CC}^n(\mathcal{A}) \to \mathcal{CC}^{n+1}(\mathcal{A})$$

for $n \geq 0$.

Definition 4.1

The space of all bounded, cyclic n-cocycles is denoted by $\mathcal{ZC}^n(\mathcal{A}, \mathcal{A}^*)$, and the subspace consisting of maps $\delta^{n-1}Q$, where Q is a bounded, cyclic $(n-1)$-cocycle, is denoted by $\mathcal{NC}^n(\mathcal{A}, \mathcal{A}^*)$. Then the continuous n^{th}-cyclic cohomology group is defined by

$$\mathcal{HC}^n(\mathcal{A}, \mathcal{A}^*) = \mathcal{ZC}^n(\mathcal{A}, \mathcal{A}^*) / \mathcal{NC}^n(\mathcal{A}, \mathcal{A}^*).$$

We take $\mathcal{HC}^0(\mathcal{A}, \mathcal{A}^*)$ to be $\mathcal{H}^0(\mathcal{A}, \mathcal{A}^*)$.

By (3.1), we see that $\mathcal{HC}^0(\mathcal{A}, \mathcal{A}^*) = \mathcal{A}^{tr}$.

In particular, the space of all bounded, cyclic derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*)$, and the set of all cyclic inner derivations from $\mathcal{A}$ to $\mathcal{A}^*$ is denoted by $\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*)$. It can be seen that every inner derivation is cyclic, and so $\mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*) = \mathcal{N}^1(\mathcal{A}, \mathcal{A}^*)$. The *first-order cyclic cohomology group* is defined by

$$\mathcal{HC}^1(\mathcal{A}, \mathcal{A}^*) = \mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*) / \mathcal{NC}^1(\mathcal{A}, \mathcal{A}^*) = \mathcal{ZC}^1(\mathcal{A}, \mathcal{A}^*) / \mathcal{N}^1(\mathcal{A}, \mathcal{A}^*).$$

Again, for example, to say that the second-order cyclic cohomology, $\mathcal{HC}^2(\mathcal{A}, \mathcal{A}^*) = \{0\}$, means that every bounded, cyclic 2-cocycle bilinear map $T: \mathcal{A} \times \mathcal{A} \to \mathcal{A}^*$ has the form $\delta^1 Q$, where $Q: \mathcal{A} \to \mathcal{A}^*$ is a bounded linear map such that

$$\langle a, Q(a) \rangle = 0 \quad (a \in \mathcal{A}).$$

In the following example, we shall show that $\mathcal{H}^2(\mathcal{A}_n, \mathcal{A}_n^*) \neq \{0\}$.

Example 4.2 *Consider the semigroup $T_n = \{e, a, a^2, \dots, a^{n-1}, a^n = o\}$. Again, set $\mathcal{A}_n = \ell^1(T_n)$, so that $\mathcal{A}_n^* = \ell^\infty(T_n)$.*

Take $n = 2$, and define the map $T: \mathcal{A}_2 \times \mathcal{A}_2 \to \mathcal{A}_2^*$ by

$$\langle \delta_z, T(\delta_x, \delta_y) \rangle = \begin{cases} 1 & if x = y = z = a \\ 0 & otherwise. \end{cases} \quad (4.5)$$

Then we *claim* that T is a *2*-cocycle but not a *2*-coboundary.

First the map T must satisfy the equation:

$$x \cdot T(y,z) - T(xy,z) + T(x,yz) - T(x,y) \cdot z = 0 \quad (x,y,z \in \mathcal{A}_2) . \quad (4.6)$$

Since $\langle \delta_a, T(\delta_a, \delta_a) \rangle = 1$, we see that $T(\delta_a, \delta_a) = \delta_a^*$ and $T(\delta_p, \delta_q) = 0$ for all other $p, q \in T_2$. We need to prove that

$$\delta_p \cdot T(\delta_q, \delta_r) - T(\delta_{pq}, \delta_r) + T(\delta_p, \delta_{qr}) - T(\delta_p, \delta_q) \cdot \delta_r = 0 \quad (4.7)$$

for all $p, q, r \in T_2$.

All four elements are zero unless at least one of the pairs (q,r) , (pq,r) , (p,qr) , and (p,q) is the pair (a,a) . Thus, there are four cases to be discussed:

Case1: Suppose that $q = r = a$. The L. H. S. of (4.7) will be equal to

$$\delta_p \cdot \delta_a^* - T(\delta_{pa}, \delta_a) + T(\delta_p, \delta_{a^2}) - T(\delta_p, \delta_a) \cdot \delta_a .$$

If $p = e$, the first two terms of (4.7) are $\delta_a^* - \delta_a^*$ and the last are zero, so (4.7) is satisfied.

If $p = a$, the terms of (4.7) are $\delta_e^* - 0 + 0 - \delta_e^*$, so (3.8) is satisfied. Lastly, if $p \neq a$ or e , then all four terms are zero and (4.7) is satisfied.

Case2: Suppose that $pq = r = a$ but $(q,r) \neq (a,a)$, so that we have $q = e$ and $p = a$. The terms of (4.7) are $\delta_e^* - 0 + 0 - \delta_e^*$, and (4.7) is satisfied.

Case3: Suppose that $p = qr = a$ but $(pq,r) \neq (a,a)$. Then $p = q = a$ and $r = e$. The terms of (4.7) are $\delta_e^* - 0 + 0 - \delta_e^*$, so (4.7) is satisfied.

Case4: If $p = q = a$, we can assume that $r \neq e$ or we are in Case3; all four terms of (4.7) are zero unless $r = a$ in which case we are back to Case1. Thus T is a 2-cocycle map.

To prove that T is not a coboundary, suppose that $T = \delta^1 Q$ for some bounded linear map $Q: \mathcal{A}_2 \to \mathcal{A}_2^*$. So from (4.5), we have

$$\begin{aligned} 0 = T(\delta_o, \delta_o) = \delta^1 Q(\delta_o, \delta_o) &= \delta_o \cdot Q(\delta_o) - Q(\delta_o) \\ &+ Q(\delta_o) \cdot \delta_o \\ &= 2\delta_o \cdot Q(\delta_o) - Q(\delta_o) . \end{aligned}$$

However, the map $\mathcal{A}_2^* \to \mathcal{A}_2^*$ such that $y \mapsto \delta_o \cdot y$ (sending δ_x^* to 0 if $x \neq o$, and $\delta_o^* + \delta_e^* + \delta_a^*$ if $x = o$) does not have $\frac{1}{2}$ as an eigenvalue. The only solution of the equation $2\delta_o \cdot Q(\delta_o) = Q(\delta_o)$ is $Q(\delta_o) = 0$. Thus $Q(\delta_o) = 0$.

Likewise,

$$\begin{aligned} 0 = T(\delta_o, \delta_a) = \delta^1 Q(\delta_o, \delta_a) &= \delta_o \cdot Q(\delta_a) - Q(\delta_o) \\ &+ Q(\delta_o) \cdot \delta_a = \delta_o \cdot Q(\delta_a) . \end{aligned}$$

So $\delta_o \cdot Q(\delta_a) = 0$, in particular $\langle Q(\delta_a), \delta_o \rangle = \langle \delta_o \cdot Q(\delta_a), \delta_o \rangle = 0$

Finally we have

$$\begin{aligned} 1 = \langle \delta_a, T(\delta_a, \delta_a) \rangle = \langle \delta_a, \delta^1 Q(\delta_a, \delta_a) \rangle &= \langle \delta_a, \delta_a \cdot Q(\delta_a) - Q(\delta_o) \\ &+ Q(\delta_a) \cdot \delta_a \rangle \\ &= 2\langle \delta_o, Q(\delta_a) \rangle - \langle \delta_a, Q(\delta_o) \rangle = 0, \end{aligned}$$

which is a contradiction. Thus T is not a 2-coboundary.

It is interesting to look at the case of this example in general. We define the map $T: \mathcal{A}_n \times \mathcal{A}_n \to \mathcal{A}_n^*$ by

$$T(\delta_p,\delta_q)=\begin{cases}0 & if\,(p,q)\neq(a,a)\\ \delta_a^* & otherwise.\end{cases}$$
(4.8)

Then we *claim* that T is a 2-cocycle but T is a 2-coboundary for $n\geq 3$.

The map T is a 2-cocycle because in the following equation:

$\delta_p\cdot T(\delta_q,\delta_r)-T(\delta_{pq},\delta_r)+T(\delta_p,\delta_{qr})-T(\delta_p,\delta_q)\cdot\delta_r=0\quad(p,q,r\in T_n)\,;$ (4.9) (3.10)

we see that all four terms in (4.9) , are zero unless at least one of the pairs (q,r) , (pq,r) , (p,qr) , and (p,q) is the pair (a,a) . Thus a similar discussing for the above four cases can be done to prove that T is a 2-cocycle.

To see that T is a 2-coboundary, let's seek a map $Q:\mathcal{A}_n\to\mathcal{A}_n^*$ such that $T=\delta^1 Q$.

From the equation (4.5) , we have

$$\begin{aligned}0=T(\delta_o,\delta_o)&=\delta^1Q(\delta_o,\delta_o)\\&=\delta_o\cdot Q(\delta_o)-Q(\delta_o)\\&\quad+Q(\delta_o)\cdot\delta_o\\&=2\delta_o\cdot Q(\delta_o)-Q(\delta_o)\,.\end{aligned}$$

However, the map $\mathcal{A}_n^*\to\mathcal{A}_n^*$ such that $y\mapsto\delta_o\cdot y$ (sending δ_x^* to 0 if $x\neq o$, and $\delta_o^*+\delta_e^*+\delta_a^*+\cdots+\delta_{a^{n-1}}^*$ if $x=o$) does not have $\frac{1}{2}$ as an eigenvalue. The only solution of the equation $2\delta_o\cdot Q(\delta_o)=Q(\delta_o)$ is $Q(\delta_o)=0$. Thus $Q(\delta_o)=0$.

Also we have

$0=T(\delta_e,\delta_e)=\delta_e\cdot Q(\delta_e)-Q(\delta_e)+Q(\delta_e)\cdot\delta_e=Q(\delta_e)$, so $Q(\delta_e)=0$.

Also we have

$0=T(\delta_o,\delta_a)=\delta_o\cdot Q(\delta_a)-Q(\delta_o)+Q(\delta_o)\cdot\delta_a$, so $\delta_o\cdot Q(\delta_a)=0$.

Suppose that $Q(\delta_a)=\lambda_0\delta_e^*+\lambda_1\delta_a^*+\cdots+\lambda_{n-1}\delta_{a^{n-1}}^*$.

We see that

$$\begin{aligned}\delta_a^*=T(\delta_a,\delta_a)&=2\delta_a\cdot Q(\delta_a)-Q(\delta_{a^2})\\&=2(\lambda_1\delta_e^*+\lambda_2\delta_a^*+\cdots\\&\quad+\lambda_{n-1}\delta_{a^{n-2}}^*)-Q(\delta_{a^2})\,,\end{aligned}$$

hence

$$\begin{aligned}Q(\delta_{a^2})&=2(\lambda_1\delta_e^*+\lambda_2\delta_a^*+\cdots+\lambda_{n-1}\delta_{a^{n-2}}^*)\\&\quad-\delta_a^*\\&=2\lambda_1\delta_e^*+(2\lambda_2-1)\delta_a^*\\&\quad+\cdots+2\lambda_{n-1}\delta_{a^{n-2}}^*\,.\end{aligned}$$

Similarly, We see that

$$\begin{aligned}0=T(\delta_a,\delta_{a^2})&=\delta_a\cdot Q(\delta_{a^2})-Q(\delta_{a^3})\\&\quad+Q(\delta_a)\cdot\delta_{a^2}\end{aligned}$$

$$=(2\lambda_2-1)\delta_e^*+2\lambda_3\delta_a^*+\cdots+2\lambda_{n-1}\delta_{a^{n-3}}^*$$

$$-Q(\delta_{a^3})+\lambda_2\delta_e^*+\lambda_3\delta_a^*+\cdots+\lambda_{n-1}\delta_{a^{n-3}}^*$$

hence

$$\begin{aligned}Q(\delta_{a^3})&=(3\lambda_2-1)\delta_e^*+3\lambda_3\delta_a^*+\cdots\\&\quad+3\lambda_{n-1}\delta_{a^{n-3}}^*\,.\end{aligned}$$

Also we see that

$$\begin{aligned}0=T(\delta_a,\delta_{a^3})&=\delta_a\cdot Q(\delta_{a^3})-Q(\delta_{a^4})\\&\quad+Q(\delta_a)\cdot\delta_{a^3}\end{aligned}$$

$$=3\lambda_3\delta_e^*+3\lambda_4\delta_a^*+\cdots+3\lambda_{n-1}\delta_{a^{n-4}}^*$$

$$-Q(\delta_{a^4})+\lambda_3\delta_e^*+\lambda_4\delta_a^*+\cdots+\lambda_{n-1}\delta_{a^{n-4}}^*$$

hence

$$\begin{aligned}Q(\delta_{a^4})&=4(\lambda_3\delta_e^*+\lambda_4\delta_a^*+\cdots\\&\quad+\lambda_{n-1}\delta_{a^{n-4}}^*)\,.\end{aligned}$$

A pattern emerge, let's look at the example when $n=3$ when we know that $Q(\delta_{a^3})=0$ so we must have $\lambda_2=\frac{1}{3}$ and the map T is a 2-coboundary for any map $Q:\mathcal{A}_3\to\mathcal{A}_3^*$ such that $T=\delta^1Q$ and $Q(\delta_o)=Q(\delta_e)=0$, $Q(\delta_a)=\lambda_o\delta_e^*+\lambda_1\delta_a^*+\frac{1}{3}\delta_{a^2}^*$, $Q(\delta_{a^2})=2\lambda_1\delta_e^*-\frac{1}{3}\delta_a^*$ and $Q(\delta_{a^3})=0$ where $\lambda_0,\lambda_1\in\mathbb{C}$.

Therefore, the map T can not be a counterexample when $n=3$.

In general, by looking at $Q(\delta_{a^k}) = 0$ for all $k \geq 3$, we must have that $Q(\delta_{a^n}) = 0$; that is $n\lambda_{n-1}\delta_e^* = 0$ so $\lambda_{n-1} = 0$ so that the map T is not a counterexample when $n \geq 3$.

5 The main result

In this section we end with our main result, where we shall reformulate the second order cohomology and cyclic cohomology groups $\mathcal{H}^{2}(\mathcal{A}_n, \mathcal{A}_n^*)$ and $\mathcal{HC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*)$ of the commutative semigroup algebra $\mathcal{A}_n$ as defined above.

For $n \in \mathbb{N}$ and from the definition of the map δ^{n} in (2.1), we form the map $\delta^{1}: \mathcal{B}^{1}(\mathcal{A}_n, \mathcal{A}_n^*) \to \mathcal{B}^{2}(\mathcal{A}_n, \mathcal{A}_n^*)$ such that for each $T: \mathcal{A}_n \to \mathcal{A}_n^*$,

we have

$$(\delta^{1}T)(a,b) = a \cdot T(b) - T(ab) + T(a) \cdot b \quad (a,b) \in \mathcal{A}_n .$$

Also, we form the map $\delta^{2}: \mathcal{B}^{2}(\mathcal{A}_n, \mathcal{A}_n^*) \to \mathcal{B}^{3}(\mathcal{A}_n, \mathcal{A}_n^*)$ such that for each $T: \mathcal{A}_n \times \mathcal{A}_n \to \mathcal{A}_n^*$, we have

$$(\delta^{2}T)(a,b,c) = a \cdot T(b,c) - T(ab,c) + T(a,bc) - T(a,b) \cdot c \quad (a,b,c) \in \mathcal{A}_n .$$

It can be shown that $\delta^{2} \circ \delta^{1} = 0$, so we can reform the second order cohomology $\mathcal{H}^{2}(\mathcal{A}_n, \mathcal{A}_n^*)$ as the following:

$$\mathcal{H}^{2}(\mathcal{A}_n, \mathcal{A}_n^*) = ker\, \delta^{2} / im\, \delta^{1} .$$

The cyclic elements of the space $\mathcal{CC}^{1}(\mathcal{A}_n, \mathcal{A}_n^*)$ are the bounded linear maps $T: \mathcal{A}_n \to \mathcal{A}_n^*$ such that

$$\langle b, T(a) \rangle = -\langle a, T(b) \rangle \quad (a, b \in \mathcal{A}_n) .$$

Also the cyclic elements of the space $\mathcal{CC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*)$ are the bounded bilinear maps $T: \mathcal{A}_n \times \mathcal{A}_n \to \mathcal{A}_n^*$ such that

$$\langle c, T(a,b) \rangle = \langle a, T(b,c) \rangle \quad (a,b,c \in \mathcal{A}_n) .$$

The map δ^{1} maps $\mathcal{CC}^{1}(\mathcal{A}_n, \mathcal{A}_n^*)$ into $\mathcal{CC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*)$.

To see that, take $T \in \mathcal{CC}^{1}(\mathcal{A}_n, \mathcal{A}_n^*)$, then for each $a, b, c \in \mathcal{A}_n$, we have

$$\begin{aligned}\langle c, T(a,b) \rangle - \langle a, T(b,c) \rangle &= \langle c, a \cdot T(b) - T(ab) + T(a) \cdot a \rangle - \langle a, b \cdot T(c) - T(bc) + T(b) \cdot c \rangle \\ &= \langle ca, T(b) \rangle - \langle c, T(ab) \rangle + \langle bc, T(a) \rangle - \langle ab, T(c) \rangle + \langle a, T(bc) \rangle \\ &\quad - \langle ca, T(b) \rangle \\ &= (\langle bc, T(a) \rangle + \langle a, T(bc) \rangle) - (\langle c, T(ab) \rangle + \langle ab, T(c) \rangle) = 0 .\end{aligned}$$

Therefore, We can reform the second order cyclic cohomology $\mathcal{HC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*)$ as the following:

$$\mathcal{HC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*) = ker\, \delta^{2} \cap \mathcal{CC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*) / \delta^{1}(\mathcal{CC}^{1}(\mathcal{A}_n, \mathcal{A}_n^*)) .$$

Finally, we conclude with our main result as presented in the following theorem:

Theorem 5.1 *Let* $\mathcal{A}_n = \ell^{1}(T_n)$, *where* $n \geq 2$. *Then*

$$\mathcal{H}^{2}(\mathcal{A}_n, \mathcal{A}_n^*) = ker\, \delta^{2} / im\, \delta^{1} .$$

and

$$\mathcal{HC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*) = ker\, \delta^{2} \cap \mathcal{CC}^{2}(\mathcal{A}_n, \mathcal{A}_n^*) / \delta^{1}(\mathcal{CC}^{1}(\mathcal{A}_n, \mathcal{A}_n^*)) . \blacksquare$$

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زمر الكوهومولوجي وزمر الكوهومولوجي الدائرية من الرتبة الثانية لبعض الجبور التبديلية لشبه زمرة

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المخلص

في هذه الورقة ، سنعيد صياغة شكل زمر الكوهومولوجي وزمر الكوهومولوجي الدائرية من الرتبة الثانية لبعض الجبور التبديلية لشبه زمرة معينة.

الكلمات المفتاحية: شبه زمرة ، زمر الكوهومولوجي وزمر الكوهومولوجي الدائرية ، جبور شبه الزمرة.